2.4 Systems identification: subspace and nonlinear methods

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Outline

What is systems identification ?

Conventional systems identification methods - ***not our focus (see 2.3 Linear Methods)***

Advanced systems identification methods - **our main focus**

- Prediction Error Methods for input-output systems
- Iterative Least-Squares
- Other Classical NL methods
- Nonlinear Models
- An industry example: valve stiction
- Subspace Identification Methods

Complementaries: Input design, data collection, model performance - **very quickly**

Identification software packages - **to do some practice**

- The Literature & our SIPPY
- Exercises: solved and proposed

Conclusions

Objectives and main ingredients of systems identification

Objectives

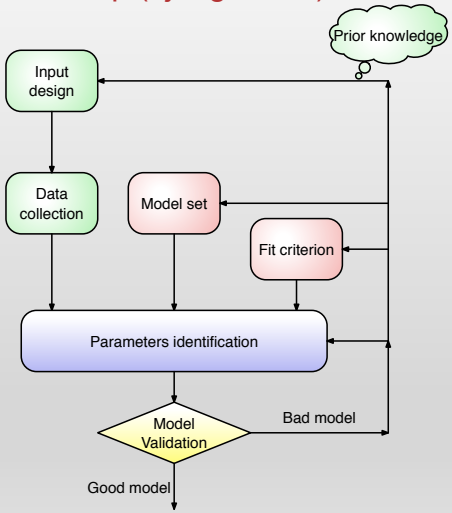
Systems identification is concerned with the **determination of a dynamic model** of the considered process given experimental (input and output) **data**

Three fundamental ingredients

1. **Data set**: Input (manipulated MVs or disturbance DVs variables) and output (CVs) can be collected during **specific** identification campaigns or during normal plant operation
2. **Model set**: A family of candidate dynamic models among which the **optimal** model will be selected
3. **Identification algorithm**: A numerical method to calculate the model parameters and obtain the optimal model

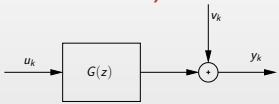
The typical procedure

The system identification loop (Ljung, 1999)



Some preliminary definitions

The reference model (in discrete-time)



Disturbance model

- By definition, the disturbance sequence $\{v\}$ is **not predictable** a priori
- Often we assume that:

$$v_k = H(z)e_k$$

in which:

- $H(z)$ is a stable Transfer Function (TF)
- e_k is zero mean, white noise with variance λ: $\lambda = \mathcal{E}\left(e_k^2\right)$

The basic relation

$$y_k = G(z)u_k + H(z)e_k$$

we want to determine two TFs: $G(z)$ and $H(z)$ given measured sequences $\{u\}$ and $\{y\}$

Some useful definitions and results

Probability quantities

- Expected Value of a random variable v:

$$\mathcal{E}(v_k) = \mathcal{E}(H(z)e_k) = \sum_{j=0}^{\infty} h_j \mathcal{E}(e_{k-j}) = 0$$

  in which $H(z) = \sum_{j=0}^{\infty} h_j z^{-j}$

- Auto-correlation of v:

$$R_v(\tau) = \mathcal{E}(v_k v_{k-\tau}) = \mathcal{E}\left(\sum_{j=0}^{\infty} h_j z^{-j} e_k \sum_{j=0}^{\infty} h_j z^{-j} e_{k-\tau}\right)$$

$$= \mathcal{E}\left(\sum_{j=\tau}^{\infty} h_j h_{j-\tau} z^{-j} e_k^2\right) = \sum_{j=\tau}^{\infty} h_j h_{j-\tau} \mathcal{E}(e_{k-j}) = \lambda \sum_{j=\tau}^{\infty} h_j h_{j-\tau}$$

- When the autocorrelation function does **not depend on** k, the signal v is said to be **stationary**

Input/output models: Linear vs. Nonlinear methods

Model structures, black-boxes and possible identification methods

Model structure	Polynomials in z		Id. Method
	$G(z)$	$H(z)$	
FIR	$B(z)$	1	Linear (e.g. **LLS**); but also NL
ARX	$A^{-1}(z)B(z)$	$A^{-1}(z)$	
ARMAX	$A^{-1}(z)B(z)$	$A^{-1}(z)C(z)$	
ARMA	1	$A^{-1}(z)C(z)$	**Nonlinear - Advanced (e.g. PEM, ILLS, RLLS)**
ARARX	$A^{-1}(z)B(z)$	$A^{-1}(z)D^{-1}(z)$	
ARARMAX	$A^{-1}(z)B(z)$	$A^{-1}(z)D^{-1}(z)C(z)$	
OE	$F^{-1}(z)B(z)$	1	
BJ (Box-Jenkins)	$F^{-1}(z)B(z)$	$D^{-1}(z)C(z)$	
GEN (Generalized)	$A^{-1}(z)F^{-1}(z)B(z)$	$A^{-1}(z)D^{-1}(z)C(z)$	

Advanced (Prediction Error) Methods: preliminaries

Motivation

- Linear methods and ARX/FIR models can be too **simplistic**, in terms of **noise description**
- **Advanced model**: increase flexibility by describing equation error $H(z)$ with a proper dynamics of the white noise e_k

Features (of PEMs)

- Given the output model $\hat{y}_k$, define the **prediction error** $\epsilon_k = y_k - \hat{y}_k$
- Often the prediction error **is filtered**: $\epsilon_k^F = L(z)\epsilon_k$, where $L(z)$ is a suitable TF which acts as a frequency filter
- Given data for N sampling times, **the loss function** is:

$$V_N = \frac{1}{N}\sum_{k=1}^{N} \mathcal{L}(\epsilon_k^F)$$

where $\mathcal{L}(\cdot)$ is a non-negative scalar function

Prediction Error Methods: details

The optimization problem

- Chosen the **model structure**, the predictor has the form: $\hat{y}_k = \varphi_k^T(\theta)\theta$
- The coefficient vector (θ) has a **nonlinear effect** in the regressor vector φ_k so that linearity is lost
- **PEM** solve an **optimization problem**:

$$\hat{\theta}_N = \arg\min_{\theta \in \mathcal{D}} V_N$$

- Depending on the choice of function $\mathcal{L}(\cdot)$ and of the model structure, the above problem can be a simple Quadratic Program (**QP**) or a more involved **NLP**
- For specific classes of model, **ad hoc** optimization algorithms were developed
- Note that these NL problems maybe **very large** for MIMO systems with evident computational issues
- Anyway, identification of input-output systems is always **MISO**

Prediction Error Methods: Model Validation

Predictors

- Given the general model: $y_k = G(z)u_k + H(z)e_k$
- the prediction error: $e_k := \epsilon_k = y_k - \hat{y}_k$

we can define:

1-step-ahead predictor

$$y_k = G(z)u_k + H(z)(y_k - \hat{y}_k) \quad \Rightarrow \quad H^{-1}(z)\hat{y}_k = G(z)u_k + (H(z) - 1)y_k$$

$$\hat{y}_k = H^{-1}(z)G(z)u_k + (1 - H^{-1}(z))y_k$$

k-step-ahead predictor

$$\hat{y}_k = W_k(z)G(z)u_k + (1 - W_k(z))y_k$$

where:

$$W_k(z) = \bar{H}_k(z)H^{-1}(z); \quad \bar{H}_k(z) = \sum_{j=0}^{k-1} h_j z^{-j}$$

being $\{h_j\}$ the coefficients of the finite impulse response of TF $H(z)$

ARMAX model

AutoRegressive Moving Average with eXternal inputs model

- The difference equation is:

$$y_k + a_1 y_{k-1} + \cdots + a_{n_a} y_{k-n_a} = b_1 u_{k-\ell-1} + \cdots + b_{n_b} u_{k-\ell-n_b} + e_k + c_1 e_{k-1} + \cdots + c_{n_c} e_{k-n_c}$$

- **noise model**: the error e_k has its own dynamics (as **moving average**)
- Polynomial form:

$$A(z) y_k = B(z) u_k + C(z) e_k$$

with:

$$C(z) = 1 + c_1 z^{-1} + c_2 z^{-2} + \cdots + c_{n_c} z^{-n_c}$$

- Observation: in this model $G(z)$ **and** $H(z)$ **have same poles**, given that:

$$G(z) = A^{-1}(z) B(z), \qquad H(z) = A^{-1}(z) C(z)$$

- ARMAX can be identified via various **nonlinear** methods, PEMs (see later...)

Other “Advanced” input/output models

Equation-Error-Type

Different **error models**:

- “**ARMA**” model:
$$A(z)y_k = C(z)e_k$$
i.e. $G(z) = 1, \quad H(z) = A^{-1}(z)C(z)$
Moving Average, but no eXternal part
- “**ARARX**” model:
$$A(z)y_k = B(z)u_k + D(z)^{-1}e_k$$
i.e. $G(z) = A^{-1}(z)B(z), \quad H(z) = A^{-1}(z)D^{-1}(z)$
a specific AutoRegressive
- “**ARARMAX**” model:
$$A(z)y_k = B(z)u_k + D(z)^{-1}C(z)e_k$$
i.e. $G(z) = A^{-1}(z)B(z), \quad H(z) = A^{-1}(z)D^{-1}(z)C(z)$
a specific ARMAX structure

Other “Advanced” input/output models

Output-Error-Type

When $G(z)$ and $H(z)$ are parametrized **independently**, no AutoRegressive part ($A(z)$) is used:

- “**Output-Error**” (OE) model:

$$y_k = F^{-1}(z)B(z)u_k + e_k$$

  i.e. $G(z) = F^{-1}(z)B(z), \quad H(z) = 1$

- “**Box-Jenkins**” (BJ) model:

$$y_k = F^{-1}(z)B(z)u_k + D^{-1}(z)C(z)e_k$$

  i.e. $G(z) = F^{-1}(z)B(z), \quad H(z) = D^{-1}(z)C(z)$
  $G(z)$ and $H(z)$ have **different poles**

General model:

$$A(z)y_k = F^{-1}(z)B(z)u_k + D^{-1}(z)C(z)e_k$$

i.e. $G(z) = A^{-1}(z)F^{-1}(z)B(z), \quad H(z) = A^{-1}(z)D^{-1}(z)C(z)$

Iterative Least-Squares for ARMAX

Preliminaries (SISO case)

- The parameter vector is: $\theta = [a_1\ a_2\ ...\ a_{n_a}\ b_1\ ...\ b_{n_b}\ c_1\ ...\ c_{n_c}]^T$, where the orders n_a,n_b,n_c are defined by the user, as for the time-delay ℓ
- The predictor is of the form: $\hat{y}_k(\theta) = \frac{B(z)}{C(z)}u_k + \left[1 - \frac{A(z)}{C(z)}\right]y_k$
- which can be rewritten as: $\hat{y}_k(\theta) = B(z)u_k + [1 - A(z)]y_k + [C(z) - 1][y_k - \hat{y}_k(\theta)]$
- Being the predictor error: $\epsilon_k = y_k - \hat{y}_k$
- the regressor vector is: $\varphi_k = [-y_{k-1}\ \ldots\ -y_{k-n_a}\ u_{k-1-\ell}\ \ldots\ u_{k-n_b-\ell}\ \epsilon_{k-1}\ \ldots\ \epsilon_{k-n_c}]^T$
- Hence, the predictor becomes: $\hat{y}_k = \varphi_k^T(\theta)\theta$
- which is not a linear regression: the terms ϵ_k can be computed only once θ is known, i.e., φ_k depends on θ. Note: we call this *pseudo-linear regression*
- An **iterative procedure** is built to get the ”best” parameters
- Easy extension to MIMO ARMAX by using a MISO approach

Iterative Least-Squares for ARMAX

The procedure (1/2)

- The whole output vector y is: $[y_M\ y_{M+1}\ \cdots\ y_N]$

  with $M = \max(n_a, n_b + \ell, n_c)$
- Regressor matrix is obtained by stacking terms for $k = M, \ldots, N$:

$$\phi = [\varphi_M\ \varphi_{M+1}\ \cdots\ \varphi_N]^T$$

- To compute parameter vector θ, ϵ sequence must be already known
- Start with an **ARX** identification and compute the first prediction error: $\epsilon = y - \phi\theta$
- Use ϵ sequence to update matrix ϕ and get a new θ by using standard **LLS**
- Go on updating the error sequence and matrix ϕ, so that, **Iterative LLS** is built
- At each step, a **norm** is computed; e.g.:

$$V_N(\theta) = \frac{1}{2(N - M + 1)} \sum_k \epsilon_k^2$$

Iterative Least-Squares for ARMAX

The procedure (2/2)

- If $V_N(\theta_{new}) < V_N(\theta_{old})$, then θ_{new} is taken to update matrix ϕ
- Otherwise a **re-evaluation** of θ is performed (*line search* method):

$$\theta^* = \lambda_s \theta_{new} + (1 - \lambda_s)\theta_{old}$$

  where $\lambda_s = \frac{1}{2^s}$, being $s = 1, 2, 3, \cdots$ the s-th step of re-evaluation
- At each step of re-evaluation:
  - norm V_N is calculated
  - if $V_N(\theta^*) < V_N(\theta_{old})$, then θ^* is taken as the next parameter vector
  - otherwise s is updated and a new re-evaluation is performed
  - when $\frac{1}{2^s}$ becomes less than $eps^{(*)}$ (the smallest representable positive number such that $1.0 + eps \neq 1.0$), procedure is stopped and θ_{old} is taken

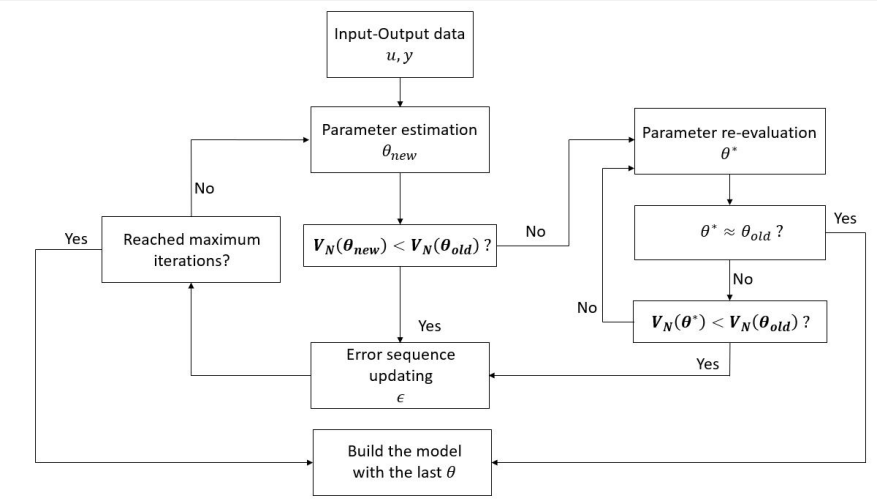
Finally, the procedure is stopped when:

1. the method finds a minimum of $V_N(\theta)$;
2. the maximum number of iterations, user defined, is reached

$^{(*)}$: $eps = 10^{-7}$ in Python 2.7, using 32 bit NumPy

Iterative Least-Squares for ARMAX

The scheme



NL Methods

Other Classical NL methods

- **Recursive Least-Squares**
  - Classical Linear Least-Squares can be **recursive**
  - a time-variant estimator allows a uniform variation of the model parameters along the identification horizon
  - A Gain Estimator, a Covariance matrix and a **Forgetting Factor** are required
  - Details are here omitted, but an example MATLAB code for ARMAX is provided
- **Nonlinear Optimization**
  - Advanced Input-Output models (e.g., BJ) may benefit from NLP
  - NL models require NL Programming
  - Also mixed methods are possible: e.g. **grid search** + LLS/PEM
  - See a later example

Nonlinear Models - EARMAX (Karra and Karim, 2009)

Extended AutoRegressive Moving Average model

- An **extended** ARMAX structure:

$$y_k = \frac{B(z)}{A(z)} u_k + \frac{C(z)}{A(z)} e_k + \frac{1}{A(z)} \eta_k$$

- not only the noise term e_k (stochastic disturbance with zero mean)
- but also a deterministic input disturbance η_k which is a **time variant** bias term representing any external non-stationary disturbances
- The model to identify is: $\hat{y}_k = \varphi_k^T(\theta)\theta + \hat{\eta}_k$
- $\{\hat{\eta}\}$ is therefore intended as a parameter which **varies slowly** over time
- Identification method must be **recursive**
- Necessary condition: build an estimator that allows a non-uniform variation of the model parameters, separating LTI part from time variant disturbance
- Parameter update with **different forgetting factors** between the two components
- An example MATLAB code is provided

Optimization Problem to Identify NL Models

Very useful implementation tools

- **Python**
  Amply validated, fast, easy-to-use, open-source, customizable

- **CasADi**
  Open-source symbolic calculation through algorithmic differentiation, numeric optimization oriented

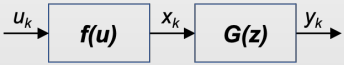
- **IPOPT**
  Standard in the class of open-source nonlinear programming (NLP) solvers

Nonlinear Block Models

In Discrete-Time

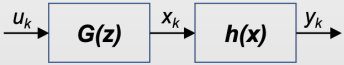
2 Blocks Models

Hammerstein



static nonlinear block followed by dynamic linear block (TF)

Wiener

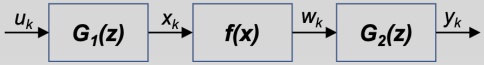


linear block (TF) followed by **static nonlinear** block

3 Blocks Models

Hammerstein - Wiener

Wiener - Hammerstein

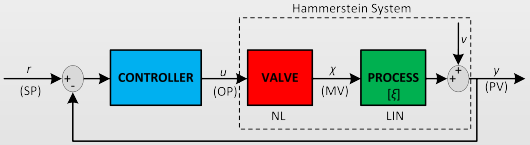


An example of NL block model: control loop with valve stiction

Extended Hammerstein SISO system

Control valve followed by the process dynamics:

- χ: valve stiction output, that is, process input
- y: process output
- u: output of a generic controller (PID or MPC)
- v: white Gaussian output noise



Whole plant dynamics

- **nonlinear** dynamics for the sticky valve, $\varphi(\cdot)$, here also **NON static**
- **linear** block for the process, SS form (**A**, **B**, **C**)

$$z_{k+1} = \begin{bmatrix} \chi_k \\ \xi_{k+1} \end{bmatrix} = \begin{bmatrix} \varphi(\chi_{k-1}, u_k) \\ \mathbf{A}\xi_k + \mathbf{B}\varphi(\chi_{k-1}, u_k) \end{bmatrix}$$

$$y_k = \mathbf{C}\xi_k + v_k$$

χ: 1^{st} component of the state vector

NL Valve Model

Smoothing function $\varphi_S(\cdot) \simeq \varphi(\cdot)$ - (Bacci di Capaci et al., 2017)

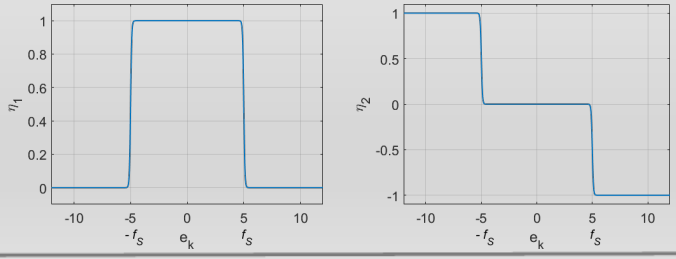
$$\chi_k = \eta_1(e_k)\chi_{k-1} + (1 - \eta_1(e_k))u_k + \eta_2(e_k)f_D$$

where

$$\eta_1(e_k) = \frac{1}{2}\tanh(\tau(e_k + f_S)) + \frac{1}{2}\tanh(\tau(-e_k + f_S))$$

$$\eta_2(e_k) = \frac{1}{2}\tanh(-\tau(e_k + f_S)) + \frac{1}{2}\tanh(\tau(-e_k + f_S))$$

- being $e_k = u_k - \chi_{k-1} \sim$ valve position error
- Tuning parameter τ: $\simeq 10^4 \Rightarrow \varphi_S(\cdot) \simeq \varphi(\cdot)$ higher value, sharper functions



Identification Problem (1/3)

Defining the Hammerstein model

Linear Process: ARX structure in discrete-time form

$$A(z)y_k = B(z)\chi_{k-\ell} + e_k$$

- $A(z), B(z)$: polynomials in backward shift operator z^{-1} (i.e. $\chi_k = z^{-1}\,\chi_{k+1}$)

$$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_{n_a} z^{-n_a}$$
$$B(z) = b_1 z^{-1-\ell} + b_2 z^{-2-\ell} + \ldots + b_{n_b} z^{-n_b-\ell}$$

- ℓ: input time-delay
- (n_a, n_b): orders on the auto-regressive and exogenous terms

Non Linear Valve: the aforesaid smoothed stiction model $\varphi_S(\cdot)$

Optimization variables X

- static and dynamic friction parameters $(\hat{f}_S, \hat{f}_D)$
- $n_a + n_b$ coefficients of ARX process model

$$X = [\hat{f}_S, \hat{f}_D, \hat{\theta}] \quad \text{with} \quad \hat{\theta} = [a_1, \ldots, a_{n_a}, b_1, \ldots, b_{n_b}]$$

Identification Problem (2/3)

One-stage nonlinear optimization problem

$$X^* = \arg\min_{f_S, f_D, \theta} \ \mathrm{SE}(y, \hat{y})$$

subject to:

$$f_{\min} \leq f_S, f_D \leq f_{\max}$$
$$f_S \geq f_D$$
$$\sigma^2(\hat{\chi}) \geq \sigma^2_{\min}$$

where

- $\hat{y} = \Phi\theta$: identified process output
- $\Phi \in \mathbb{R}^{N \times n_a + n_b}$: regressor matrix of the measurements (u,y)
- N: number of data points
- $\hat{\chi}$: identified valve position

Remarks

- Square Error (SE) objective function: $\mathrm{SE}(y, \hat{y}) = \frac{1}{2}(y - \hat{y})^T(y - \hat{y})$
- Constraint on the variance $\sigma^2(\hat{\chi})$: **valve is forced to oscillate** due to the presence of stiction, e.g. a safe choice $\sigma^2_{\min} = 0.1\sigma^2(u)$
- Time-delay ℓ and model orders are assumed as parameters

Identification Problem (3/3)

Initialization

Suitable initial point X_0:

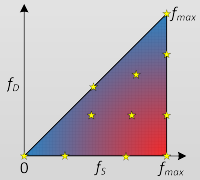
- $\hat{\theta}_0$: ARX model identification, **valve stiction free**

$$\hat{\theta}_0 = (\phi_0)^+ y = [\phi_0^T \phi_0]^{-1} \phi_0^T y$$

ϕ_0: initial regressor matrix computed by stacking linear regressor vectors $\varphi_{0,k}$ $\forall$ time sample k

$$\varphi_{0,k} = [-y_{k-1}, ..., -y_{k-n_a}, u_{k-1-t_d}, ..., u_{k-n_b-\ell}]$$

- $\hat{f}_{S,0}, \hat{f}_{D,0}$: define a **triangular shape domain**
  - $f_{\max} \geq f_S \geq f_D \geq f_{\min} = 0$
  - $f_{\max} = \Delta u$, oscillation span of controller output
  - $f_S + f_D = \Delta u$, square-shaped signal



Multiple starts (M) to avoid to be stuck in a local minimum
Start from the domain boundaries, step $\Delta f_S = \Delta f_D = 0.5$
Choose the **best** solution in terms of **objective function and infeasibility**

Subspace Identification Methods (SIM): introduction

Motivations

- Multivariable input-output systems identification requires **prior knowledge** or **trial-and-error** to determine the system orders (note: **Information Criteria** can be used; see complements...)
- Input-output systems identification is always **MISO**, whereas in some cases it would desirable to **directly** identify **MIMO** models
- Identification of **advanced multivariable** models (e.g., ARMAX, OE, etc.) may require solution of large **nonconvex nonlinear programming** problems

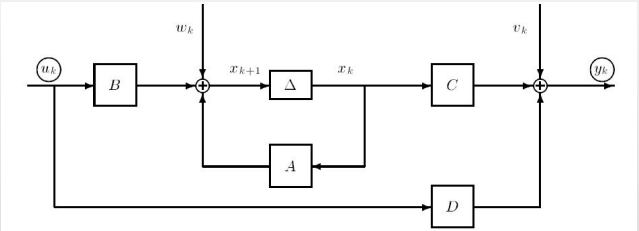
Features

- **Direct** identification of a DLTI state-space model
- Applicable to both **MIMO** and MISO approaches
- **Compact** multivariable state-space representation
- Very **little prior knowledge** required (an **upper bound** to the order)
- Based on **reliable linear algebra** decompositions

State-space systems: basic definitions

DLTI system: **process** form

$$x_{k+1} = Ax_k + Bu_k + w_k$$
$$y_k = Cx_k + Du_k + v_k$$



- dimensions: $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$
- note: n often **unknown**; m, p known; usually $D = 0$, for physical systems
- **noise** term: split into *measurement* v_k and *process* w_k noise
- Assumption: $\{w\}, \{v\}$ are sequences of independent random variables with zero mean and covariances:

$$\mathcal{E}(w_k w_k^T) = Q \quad \mathcal{E}(w_k v_k^T) = S \quad \mathcal{E}(v_k v_k^T) = R$$

State-space systems: basic definitions

DLTI system: **innovation** and **predictor** forms

Innovation form:

$$x_{k+1} = Ax_k + Bu_k + Ke_k$$
$$y_k = Cx_k + e_k$$

Predictor form ($A_K = A - KC$):

$$x_{k+1} = A_K x_k + Bu_k + Ky_k$$
$$y_k = Cx_k + e_k$$

where K is the steady-state **Kalman filter gain**, obtained from Algebraic Riccati Equation

Main assumptions

- (A, B) controllable; (A, C) observable; $A_K = A - KC$ **strictly Schur** ($|\lambda_i(A_k)| < 1,\ \forall\, i$)
- The innovation $\{e_k\}$ is a **stationary**, zero mean, white noise process:

$$\mathcal{E}(e_j e_j^\top) = R_e, \qquad \mathcal{E}(e_i e_j^\top) = 0 \quad \text{for } i \neq j$$

- Input $\{u_k\}$ and output $\{y_k\}$ data sequences are available for $k = 0, \ldots, N$

Indirect routes to get this LTI model

- They can be obtained **via realization** of input-output models (TF $\Rightarrow$ SS)
- Often the obtained **order** is quite **high**, with **no** perceivable **advantages**

SIM: classification (Qin, 2006)

Traditional - N4SID, MOESP, CVA

- later grouped into a unifying theorem (Overschee and Moor, 1995)
- estimate system matrices $(A, B, C, (D))$ from the **process** form
- seen as a **singular value decomposition** (SVD) of a suitable weighted matrix
- basic assumption: u_k and w_k, v_k are uncorrelated $\Rightarrow$ may **fail on closed-loop** data
- **do not enforce** any block-triangular matrix and then casual models

Parsimoniuos - PARSIM-S, PARSIM-P, PARSIM-K, PARSIM-E

- estimate system matrices $(A, B, C, (D), K)$ from the **innovation/predictor** form
- remove non-causal terms in the linear projections by enforcing **causal models**; in particular, enforce the lower triangular structure of matrix H_r^u
  - sequential **PARSIM-S** (Qin et al., 2005) and parallel **PARSIM-P** (Qin and Ljung, 2003)
    - oldest and most common algorithms
    - basic assumption: u_k and e_k are uncorrelated $\Rightarrow$ **still may fail on CL** data
  - **PARSIM-K** (Pannocchia and Calosi, 2010) and **PARSIM-E** algorithms (Hou et al., 2015):
    - specifically consistent with CL data

Basic SIM: algorithm derivation

An r-step prediction model

- For each k, define an r-step prediction model (here, innovation form):

$$y_k = Cx_k + e_k;\;\; y_{k+1} = Cx_{k+1} + e_{k+1} = CAx_k + CBu_k + CKe_k + e_{k+1};\;\; y_{k+2} = \dots$$

$$\underbrace{\begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ \vdots \\ y_{k+r-1} \end{bmatrix}}_{\bar{y}_k} = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{r-1} \end{bmatrix}}_{\Gamma_r} x_k + \underbrace{\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ CB & 0 & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{r-2}B & CA^{r-3}B & \cdots & 0 \end{bmatrix}}_{H_r^u} \underbrace{\begin{bmatrix} u_k \\ u_{k+1} \\ u_{k+2} \\ \vdots \\ u_{k+r-1} \end{bmatrix}}_{\bar{u}_k}$$

$$+ \underbrace{\begin{bmatrix} I & \cdots & \cdots & 0 \\ CK & I & \cdots & 0 \\ CAK & CK & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{r-2}K & CA^{r-3}K & \cdots & I \end{bmatrix}}_{H_r^e} \underbrace{\begin{bmatrix} e_k \\ e_{k+1} \\ e_{k+2} \\ \vdots \\ e_{k+r-1} \end{bmatrix}}_{\bar{e}_k}$$

- **Repeat** for $k \in \{r, \dots, M = N - r + 1\}$ and **concatenate horizontally**:

$$\underbrace{\begin{bmatrix} \bar{y}_r & \cdots & \bar{y}_M \end{bmatrix}}_{Y} = \Gamma_r \underbrace{\begin{bmatrix} x_r & \cdots & x_M \end{bmatrix}}_{x} + H_r^u \underbrace{\begin{bmatrix} \bar{u}_r & \cdots & \bar{u}_M \end{bmatrix}}_{U} + H_r^e \underbrace{\begin{bmatrix} \bar{e}_r & \cdots & \bar{e}_M \end{bmatrix}}_{E}$$

Basic SIM: algorithm - extended observability matrix

The basic relation

- The previous relation is written **compactly** as:

$$Y = \Gamma_r x + H_r^u U + H_r^e E$$

- Γ_r is called **extended observability matrix**
- H_r^u and H_r^e are **lower triangular block Toeplitz** matrices

Computing the extended observability matrix

- Note: the **different methods** basically **separate here**.
  E.g., in PARSIM-K, express:

$$X = \begin{bmatrix} x_r & \cdots & x_M \end{bmatrix} = A_K^r \begin{bmatrix} x_0 & \cdots & x_{M-r} \end{bmatrix} + \begin{bmatrix} A_K^{r-1} B & \cdots B \end{bmatrix} U_p + \begin{bmatrix} A_K^{r-1} K & \cdots K \end{bmatrix} Y_p$$

$$\approxeq \underbrace{\begin{bmatrix} A_K^{r-1} B & \cdots B & A_K^{r-1} K & \cdots K \end{bmatrix}}_{L_z} \underbrace{\begin{bmatrix} U_p \\ Y_p \end{bmatrix}}_{Z_p}$$

- **Solve** the basic relation: $Y = \Gamma_r L_z Z_p + H_r^u U + H_r^e E$ to obtain $(\Gamma_r L_z)$ from **LS**
- Compute Γ_r, having n columns with $n < r$, from a **truncated SVD** of $(\Gamma_r L_z)$

Basic SIM: algorithm - compute (A, C)

Computing (A, C): another LS problem

- After Γ_r is obtained, we can easily compute C (MATLAB notation):

$$C = \Gamma_r(1 : p, :)$$

that is, first p rows of Γ_r, being p the output number

- The matrix A is instead obtained from the **LS problem**, by using shift-invariance property of Γ_r:

$$\Gamma_r(p + 1 : p \times r, :) = \Gamma_r(1 : p \times (r - 1), :)A$$

- Hence, the solution is:

$$A = \Gamma_r(1 : p \times (r - 1), :)^{+}\Gamma_r(p + 1 : p \times r, :)$$

where $\{\cdot\}^{+}$ is the pseudo-inverse matrix; $\Gamma_r(1 : p \times (r - 1), :)$ and $\Gamma_r(p + 1 : p \times r, :)$ are Γ_r without the last and the first p rows, respectively

Basic SIM: algorithm - compute B, (D), and x_0

Obtaining B and x_0 via LS

- From: $x_{k+1} = Ax_k + Bu_k + x_0; \quad \hat{y}_k = Cx_k$
- Being $x_{k+1} = zx_k$, obtain: $x_k = (zI - A)^{-1}(Bu_k + x_0)$
- With (A, C) known, we can write a **linear predictor**:

$$\hat{y}_k = C(zI - A)^{-1}Bu_k + C(zI - A)^{-1}x_0 = \varphi_k\theta, \qquad \text{with } \theta = \begin{bmatrix} \text{Vec}(B) \\ x_0 \end{bmatrix}$$

  where $\text{Vec}(B)$ is vectorized B matrix along the rows (easy extension when $D \neq 0$)

- repeating until N and stacking:

$$Y = \phi\theta$$

  where

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \qquad \phi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{bmatrix}$$

- Solve an **LS problem**:

$$\theta = (\phi^\top\phi)^{-1}\phi^\top Y$$

Identification software packages

Commercial packages

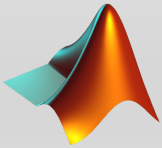
Many solutions from commercial vendors, covering the various identification methods

- **System Identification Toolbox ™** in **MATLAB** (MathWorks, 2021): the most famous and consolidated package
- Other options:
  - ISIAC software (Tona and Bader, 2006)
  - NI LabVIEW System Identification Toolkit (Instruments, 2021)

Open-source packages

Many examples, written on different programming languages, as the various **MATLAB-based** toolboxes:

- UNIT (Ninness et al., 2013)
- CONTSID (Garnier et al., 2012)
- CAPTAIN (Young and Taylor, 2012)
- ITSIE (Guzmán et al., 2012)



SIPPY: System Identification Package for PYthon

Main features

One of **most complete** open-source package for **Python** – to the best of our knowledge ...

- covering a wide range of identification methods
- possibility to identify **multivariable** systems
- focused only on **linear** models
- excluded nonlinear systems
  – see other software, e.g. NL-ARX (MathWorks, 2021) or Hammerstein-Wiener models (Ninness et al., 2013)

Models & Data

- identifies both **input-output** and **state space** models
- uses input-output data (Open and Closed Loop)
  for a general multivariable system with m inputs and p outputs:

$$\mathbf{u} = [u_0 \; u_1 \; u_2 \; \dots \; u_{N-1}], \quad \mathbf{y} = [y_0 \; y_1 \; y_2 \; \dots \; y_{N-1}]$$

where N: number of samples

SIPPY: some details on state space models

Algorithms implemented

- **modified N4SID**: improved "*combined algorithm 2*" (Overschee and Moor, 1996)
- **original aspect** of **SIPPY**: include 3 Parsimonious methods
- **truncated SVD**:
  - retaining up to the $n-$th singular value
  - alternatively, specify a threshold value TV with the maximum order allowed
  - otherwise, employ an information criteria
- **Extensive simulation** studies:
  $\Rightarrow$ superior performance in terms of the variance of the residuals
- **lower computation load** as only a single state sequence is identified

User specifications and choices

- model orders and horizons (future and past)
- direct input-output relation, i.e., matrix $D \neq 0$
- matrix A is analyzed with a **stability test**
  for *traditional* methods, impose the stability of matrix A: $\rho(A) \equiv \max_i(|\lambda_i|) < 1$

Summary about **SIPPY**

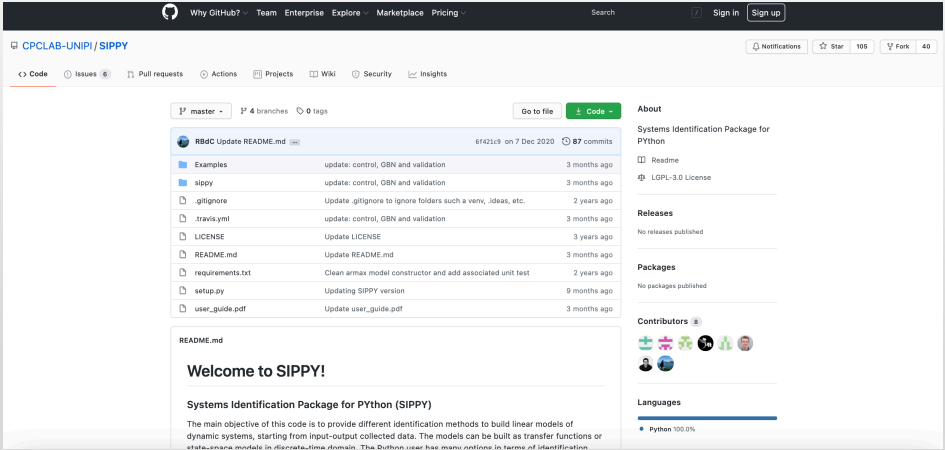
- An **open-source package for Python** with various **sys id** structures and methods
- **Input-output** models in 9 structures and 3 algorithms (*):
  - FIR, ARX $\Rightarrow$ LLS
  - ARMAX $\Rightarrow$ I-LLS
  - ARMAX, ARARX, ARARMAX, OE, BJ, GENERALIZED $\Rightarrow$ NLP (*: next release)
- **State space** models, 6 algorithms:
  - 3 standard approaches (N4SID, MOESP, and CVA)
  - 3 parsimonious (PARSIM-S, PARSIM-P, PARSIM-K)
- 3 **information criteria**: to help the choice of suitable model orders when not known a-priori
- Several **file examples** included: pure numerical and simulation
- The state-of-art **MATLAB toolbox**: taken as **reference to test** performance and accuracy
- **Comparable results** vs. MATLAB: but higher efficiency in terms of computational times

- Underlying **code**:
  - default settings: intended to be simple for beginners
  - user options: to set algorithms parameters
  - **repository**: freely available at **GITHUB**

GitHub

The GitHub Repository

Open-source access: *https://github.com/CPCLAB-UNIPI/SIPPY*



Simulation Examples – solved #1

The plant: a MIMO ARMAX system ($m = 4, p = 3$)

	Output 1	Output 2	Output 3
Input 1	$g_{11} = \frac{4z^3+3.3z^2}{z^5-0.3z^4-0.25z^3-0.021z^2}$	$g_{21} = \frac{-85z^2-57.5z-27.7}{z^4-0.4z^3}$	$g_{31} = \frac{0.2z^3}{z^4-0.1z^3-0.3z^2}$
Input 2	$g_{12} = \frac{10z^2}{z^5-0.3z^4-0.25z^3-0.021z^2}$	$g_{22} = \frac{71z+12.3}{z^4-0.4z^3}$	$g_{32} = \frac{0.821z^2+0.432z}{z^4-0.1z^3-0.3z^2}$
Input 3	$g_{13} = \frac{7z^2+5.5z+2.2}{z^5-0.3z^4-0.25z^3-0.021z^2}$	$g_{23} = \frac{-0.1z^3}{z^4-0.4z^3}$	$g_{33} = \frac{0.1z^3}{z^4-0.1z^3-0.3z^2}$
Input 4	$g_{14} = \frac{-0.9z^3-0.11z^2}{z^5-0.3z^4-0.25z^3-0.021z^2}$	$g_{24} = \frac{0.994z^3}{z^4-0.4z^3}$	$g_{34} = \frac{0.891z+0.223}{z^4-0.1z^3-0.3z^2}$
Error model	$h_1 = \frac{z^5+0.85z^4+0.32z^3}{z^5-0.3z^4-0.25z^3-0.021z^2}$	$h_2 = \frac{z^4}{z^4-0.4z^3}$	$h_3 = \frac{z^4+0.7z^3+0.485z^2+0.22z}{z^4-0.1z^3-0.3z^2}$

orders: p for output and error, $p \times m$ for inputs and time-delays:

$$n_a = [3\ 1\ 2], \quad n_b = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 3 & 2 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}, \quad n_c = [2\ 0\ 3], \quad \Theta = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

Parameters & Settings

- **Input Data**: 4 independent GBN with a switch probability equal to 3%
- **Models tested**: ARX and ARMAX, with **known** system orders and time-delays
- **White noise** $e^{(i)}$: 5 levels with different variances σ^2
- **Monte Carlo simulations**: $\forall$ noise level, a set of simulations with $N = 400$ data
  500 simulations for the **identification** stage and 500 for the **validation**

Simulation Examples – solved #1

Identification performance index

Performance evaluated by using the **explained variance** (**EV**):

$$EV^{(i)} = 1 - \frac{\sum_k (\hat{\epsilon}_k^{(i)})^2}{\sum_k (y_k^{(i)} - \overline{y}^{(i)})^2}$$

where $\overline{y}^{(i)}$ is the mean value of the i-th output

When a model returns $EV^{(i)} < 0$ for an output, the EV is considered equal to zero

Results & Discussion

Average explained variance $\overline{EV}$,
for identification **ID** and validation **VA** data sets

	σ_1^2	σ_2^2	σ_3^2	σ_4^2	σ_5^2
ARMAX (ID)	*0.9423*	*0.9585*	*0.9513*	*0.9386*	*0.9293*
ARX (ID)	*0.9955*	*0.9918*	*0.9858*	*0.9769*	*0.9672*
ARMAX (VA)	*0.9417*	*0.9565*	*0.9474*	*0.9341*	*0.9230*
ARX (VA)	*0.9940*	*0.9890*	*0.9819*	*0.9712*	*0.9622*

- **ARX**: shows superior performance
- **ARMAX**:

$EV < 0$ for several simulations,
which make values of $\overline{EV} \Downarrow$

whether the identification is successful,
single EV is usually greater than the one
obtained by ARX

note: **this is NOT a general result !!!**

Simulation Examples – solved #2

Plant & Models

- **MIMO State-Space** system ($m = 2, p = 2, n = 3$)
- *White noise*: 4 different levels, with the same variance σ^2 for both outputs
- **Model tested**: N4SID and PARSIM-K ($f_h = p_h = 20$)

Parameters & Settings

- **Closed-loop CL mode**: data collected ($\mathbf{u}, \mathbf{y}$), with known model order
  - *reference vector* r: 2 independent GBNs with a switch probability of 2%
  - *CL input*: proportional control law $u_k = K_c(r_k - y_k)$
- **Monte Carlo simulations**: $\forall$ noise level, a set of simulations with $L = 500$
  - 500 for the *identification* stage and 500 for the *validation* stage

Identification performance index

Performance evaluated with the explained variance (EV): $EV^{(i)} = 1 - \frac{\sum_k (\epsilon_k^{(i)})^2}{\sum_k (y_k^{(i)} - \overline{y}^{(i)})^2}$

where $\overline{y}^{(i)}$: mean value of the i-th output

when a model returns $EV^{(i)} < 0$, the EV is considered equal to zero

Simulation Example – solved #2

Results & Discussion

Overall: mean variance $\overline{EV}$
for identification (ID) and validation (VA) data sets

	σ_1^2	σ_2^2	σ_3^2	σ_4^2
N4SID (ID)	*0.9988*	*0.9859*	*0.8793*	*0.4256*
PARSIM-K (ID)	*0.9989*	*0.9886*	*0.8825*	*0.4089*
N4SID (VA)	*0.9988*	*0.9871*	*0.8766*	*0.4181*
PARSIM-K (VA)	*0.9988*	*0.9881*	*0.8758*	*0.3972*

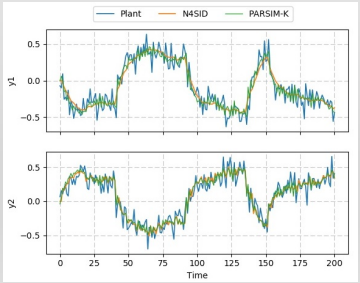
A generic simulation example

Validation stage with noise variance σ_3^2

- both **PARSIM-K and N4SID**: excellent performance
- proper fitting of the original output
- despite being a hard case of CL data

PARSIM-K shows:

- better performance in the identification stage
- a slightly lower robustness to noise in the validation stage



Simulation Examples – proposed

Choose among the simulation examples included in SIPPY

- pure numerical cases: ARX SISO, ARMAX SISO, SS
- Wood-berry column (*)
- Continuos Stirred Tank
- Continuos Stirred Tank Reactor (*)
- Triple effect evaporators (*)
- .. but also build your own examples and/or use your own data

Plug and Play

- test different models structures and algorithms
- test parameters sensitivity: model orders, noise levels, input design ...

Conclusions

Some observations

- Systems identification is of **primary importance** for the success (or the failure) of **advanced control**
- Since its origin in the process industries (identification of FIR models via least-squares), many advances were made:
  - The importance of data collection has been recognized and widely accepted
  - Robust and efficient identification methods have been developed (e.g., LS, ILS, RLS, QP, NLP)
  - Especially for MPC design, subspace identification methods has been very popular:
    - traditional: N4SID, MOESP, CVA;
    - parsimonious: PARSIM-S, -P, -K, -E.
- Examples with software **SIPPY**: Systems Identification Package for PYthon
  `https://github.com/CPCLAB-UNIPI/SIPPY`

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